1 Stress fields around a crack tip

Let's consider a linear elastic material subjected to a Mode I and a Mode II loading. The stress field around the crack tip is given by the expression in Table 2.1 of Fracture Mechanics : Fundamentals and Applications, 4th edition, by T.L. Anderson. (You can access the book online through the EPFL library). For the following exercises, consider the stress intensity factors $K_I = K_{II} = 1$, $0.001 < r < 0.3$ and $-\pi < \theta < \pi$.

Question 1

Using the stress field expressions, plot the σ_{xx}, σ_{yy} and τ_{xy} stresses using matlab or Python, respectively for Mode I and Mode II loading. Use a logarithmic scale for the colorbar.

Question 2

Plot the Von Mises stress using the stress field expressions for Mode I and Mode II. As a reminder, for plane stress, the Von Mises stress is given by:

$$\sigma_{\text{VM}} = \sqrt{\sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2 + 3\tau_{xy}^2} \tag{1}$$

2 Relation between stress intensity factor K and energy release rate G

Irwin's singular solution is consider as a pillar of linear elastic fracture mechanics (LEFM) since he connected this near-tip approximation described by K to the elastic energy release by crack advance and Griffith energy balance, the other pillar of LEFM. In this exercise, we will reproduce the demonstration of Iwrin. Let us first consider a through crack loaded in Mode-I in a plate (plane stress condition). We will study the infinitesimal extension Δa of the crack of length $2a$ as illustrated in Figure 1.

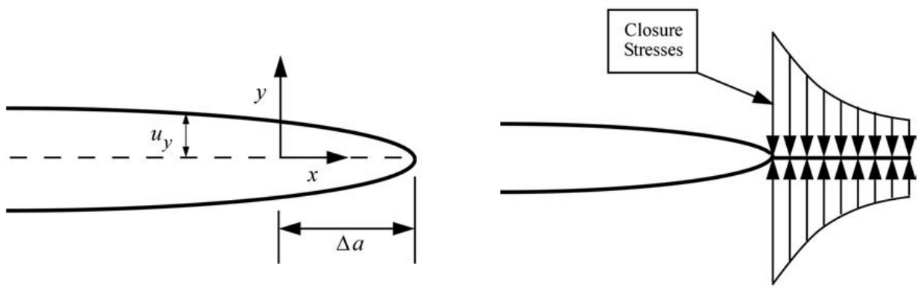

Figure 1: Irwin computed the work required to close a crack along Δa.

Question 1

Starting from the expression of the total energy of the system U

$$U(a) = \Pi(a) + W_s(a), \tag{2}$$

recall the definition of the energy release rate G and the fracture energy G_c proposed by Irwin.

Question 2

To estimate the elastic energy release by a crack advance Δa (being equal by equilibrium to the work required to open the two surfaces), Irwin considered the opposite action and compute the work to close back the crack over an infinitely small distance Δa. Using the displacement field $u_y(a + \Delta a)$ and the stress field $\sigma_{yy}(a)$, compute the work $W_{close}(\Delta a)$ required to close the crack faces over Δa. Use the near-tip singular fields given by Irwin's approximation:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos(\frac{\theta}{2}) \left[1 + \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2})\right] \tag{3}$$

$$u_y = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin(\frac{\theta}{2}) \left[\kappa + 1 - 2\cos^2(\frac{\theta}{2})\right], \tag{4}$$

r and θ being the polar coordinates originated from crack tip, μ the shear modulus and $\kappa = 3 - 4\nu$ for plane strain.

Hint: In absence of body forces, Clapeyron's theorem states that:

$$\int_V \boldsymbol{\sigma\epsilon} \, dV = \int_A \boldsymbol{fu} \, dA, \tag{5}$$

$\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ being respectively the elastic stress and strain tensor within the continuum volume and $\boldsymbol{u}$ the displacements at boundaries associated with the external forces.

Question 3

Take the limit of Δa being zero and compute the resulting energy release rate as

$$G = \lim_{\Delta a \to 0} \left[\frac{W_{close}(\Delta a)}{\Delta a}\right]. \tag{6}$$

3 Crack in a steel vessel

Consider a spherical steel pressure vessel whose radius is 1 m and whose wall thickness is 10 mm, designed to work under a maximum internal pressure of 10 MPa (100 bars or about 1500 psi).

Question 1

You discover that the pressure vessel walls contain semi-circular edge-cracks of radius $a_0 = 1$mm. If the particular grade of steel has a plane stress critical stress intensity factor of 20 MPa $\cdot$ m$^{0.5}$, do you think that the pressure vessel is safe to use?
Hint : For a short (1 mm << 10 mm) semi-circular edge-crack you can approximate the cylinder walls as a sheet (the crack is much small than the radius of curvature of the walls) and use:

$$K_I = 1.12 \frac{\sigma\sqrt{\pi a}}{\frac{3\pi}{8} + \frac{a^2\pi}{8c^2}} (\sin^2\theta + \frac{a^2}{c^2}\cos^2\theta)^{1/4} \tag{7}$$

With $a = c = a_0$.

Question 2

The Young's modulus of the steel is 200 GPa. Assuming plane stress conditions, what is G_c?

Question 3

Thanks to a secret additive, the R-curve behaviour of the steel takes the form:

$$R = G_c(1 + \frac{a - a_0}{\zeta}) \tag{8}$$

for a crack length $a > a_0$ where $\zeta = 5 \cdot 10^{-4}$ m. Do you think the pressure vessel will leak or explode if it is subjected to an indefinite increase in pressure ?